On a Standard of Mutual Inductance.

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(From the National Physical Laboratory.)

CONTENTS.

§ 1. *Introductory.*—In many electrical measurements, such as those of capacity and inductance, as well as in the magnetic testing of iron, an accurately known standard of mutual inductance is of great value. It is sometimes convenient to derive such a standard from the standard unit of resistance, and this may be done in several ways, for example, by the well-known method of the ballistic galvanometer; or by Carey Foster's method the mutual inductions may be tested against a condenser whose capacity has been found in terms of resistance and frequency by Maxwell's commutator method; or it may be obtained directly in similar terms by the help of an unknown inductance by the Hughes-Rayleigh method.

In the National Physical Laboratory I have used both of these latter methods (with the help of a vibration galvanometer) to obtain a working standard of mutual inductance.* But this procedure is somewhat illogical, seeing that the unit of resistance has been itself commonly determined by the aid of mutual inductances calculated from the dimension of the coils or other conductors used; thus for the highest accuracy it is desirable to revert to a standard whose value can be determined solely from the geometrical dimensions. Accordingly, some eighteen months ago, I took in hand the investigation of a suitable design for such a standard, and I proceed to describe the result at which I arrived.

§ 2. *Practical Conditions.*—The most important conditions governing the design were that the standard must be—

(*a*) accurately calculable from the geometrical dimensions;

(*b*) as permanent as possible; and

* 'Phys. Soc.,' May, 1907.

(*c*) of a value sufficiently large to give high sensitivity when used in comparison methods, such as those mentioned above.

In addition to these, it was desirable to keep the resistances of the parts as low as possible and to avoid (as far as possible) the occurrence of eddy currents, and also of capacity effects between the primary and secondary circuits. As is so often the case in designs, a compromise had to be effected between the various conditions, and so it was decided to make the value approximately 0·01 henry.

In order to carry out some of the above conditions, it is clearly desirable that the distance from the primary circuit to the secondary should, for all points, be relatively as great as possible. For convenience, let us call the circuit with the smaller number (n_1) of turns the primary, the secondary having n_2 turns, and let M be the mutual inductance in millihenries. Since M, for a given geometrical disposition of the circuits, is proportional to $n_1 n_2$, a little consideration showed that for $M = 10$ millihenries it would be desirable to make $n_1 n_2$ of the order of 100,000. Now it seems to be generally recognised that, for a coil whose dimensions have to be accurately measured, the satisfactory construction is of bare wire wound in an accurate screw-cut on a marble cylinder. If the above conditions be kept in mind, it is out of the question, with $n_1 n_2 = 100{,}000$, to attempt to construct both the primary and secondary circuits of single-layer coils.

As will be shown later, the solution of the problem consists in making the primary an accurately measured single-layer coil or coils, while the secondary is a many-layered coil, *so designed that its dimensions and position do not require to be very accurately determined.* The possibility of such a design and the method of carrying it out were found by an examination of the manner in which M varies with the dimensions and positions of the primary and secondary coils.

I proceed to give some of the more interesting results of this examination.

§ 3. *Variation of* M *with Dimensions and Positions of the Coils.*—First, let us consider the simple case of two circular co-axial coils with winding of negligible section, assuming for convenience of calculation $n_1 n_2 = 100{,}000$. Let their radii be A and a, and let b be the distance between their centres.

Let $\quad k = 2\sqrt{\mathrm{A}a}/\sqrt{(\mathrm{A}+a)^2+b^2} = \sin\gamma, \quad$ and $\quad k' = \cos\gamma.$

Then* (in henries)

$$\mathrm{M} = -4\pi n_1 n_2 \sqrt{\mathrm{A}a}\left\{\left(k-\frac{2}{k}\right)\mathrm{F}_1+\frac{2}{k}\mathrm{E}_1\right\}10^{-9}, \tag{1}$$

where E_1 and F_1 are complete elliptic integrals to modulus k.

* Maxwell's 'Elect. and Mag.,' vol. 2, § 701.

Since
$$\frac{dE_1}{dk} = \frac{1}{k}(E_1 - F_2),$$
and
$$\frac{dF_1}{dk} = \frac{1}{kk'}(E_1 - k'^2 F_1),$$
it can be shown that
$$\frac{\partial M}{\partial a} = \frac{2\pi n_1 n_2}{\sqrt{(A+a)^2+b^2}}\left\{2a(F_1 - E_1) + (A-a)\left(\frac{k}{k'}\right)^2 E_1\right\} 10^{-9}. \qquad (2)$$

From equation (1), mainly by the help of Lord Rayleigh's table,* and in some cases by Nagaoka's formula and tables,† were calculated sets of values of M for a fixed value of A (viz., 10 cm.), and varying values of a and b.

From these calculations the families of curves shown in figs. 1 and 2 were drawn. Each curve in fig. 1 corresponds to a constant value of a, and in

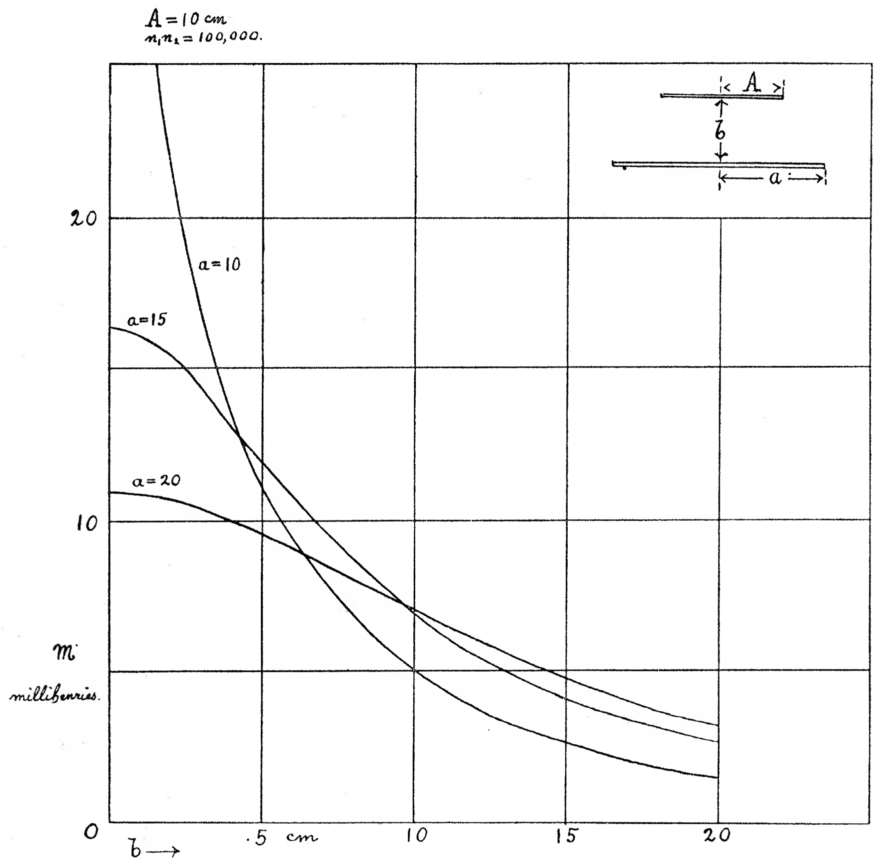

Fig. 1.

* *Ibid.*, 2nd edition.
† Tōkyō Sūg, 'Būtsu. Kiji-gaiyo,' vol 2, No. 17.

fig. 2 to a constant value of b. It will be noticed that in fig. 1 M is always maximum (and $\partial M/\partial b = 0$) only when $b = 0$, while in fig. 2 the maximum value occurs for a different value of a (not zero) for each curve. Thus when only two coils are used, b should either be zero or relatively large. When b is zero, $\partial M/\partial a$ is zero only when $a = A$ or the coils coincide; when b is relatively large, the whole construction has to be very bulky in order to obtain a large enough M. Accordingly the case of two coils is not sufficient for our purpose.

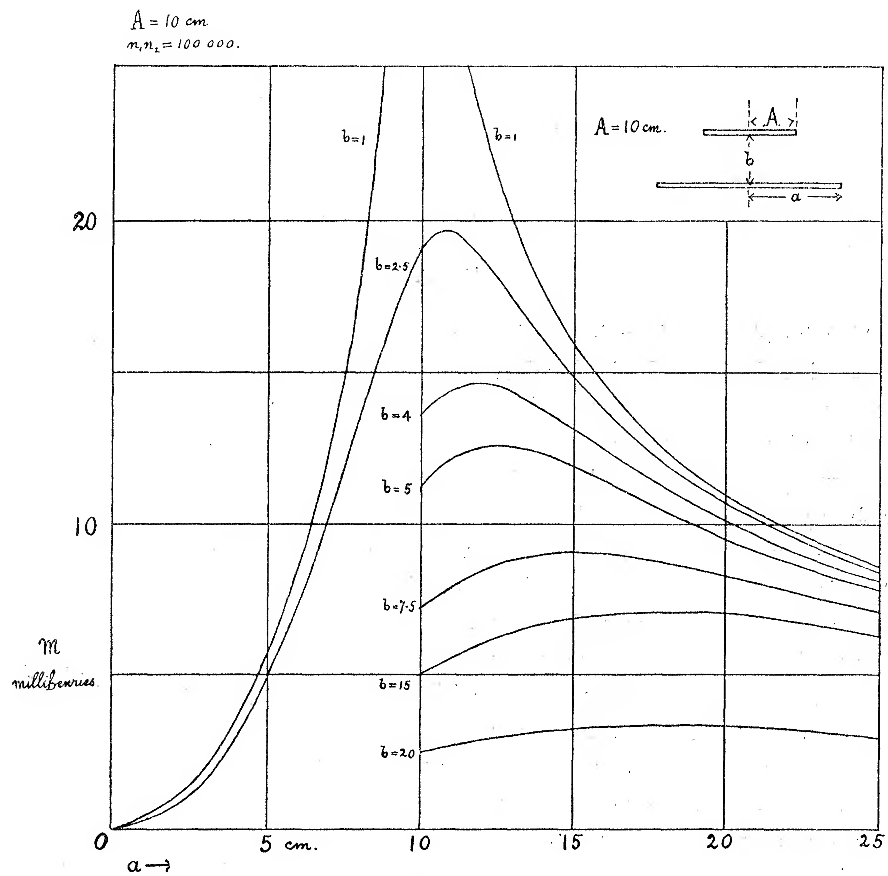

FIG. 2.

If, however, the primary consists of two equal coils arranged with the secondary between them as in fig. 3, all three being coaxial, M is a maximum or minimum for axial displacements when $b_1 = b_2$.

If, then, for any desired value of b ($= b_1 = b_2$) we choose from the proper curve in fig. 2 the value of a which gives M a maximum, the mutual inductance thus obtained varies only very slightly for small variations of a

or small axial displacements of the centre coil; in fact, we have placed the secondary coil in such a position that all round its mean circumference *the field due to the primary coils is zero.*

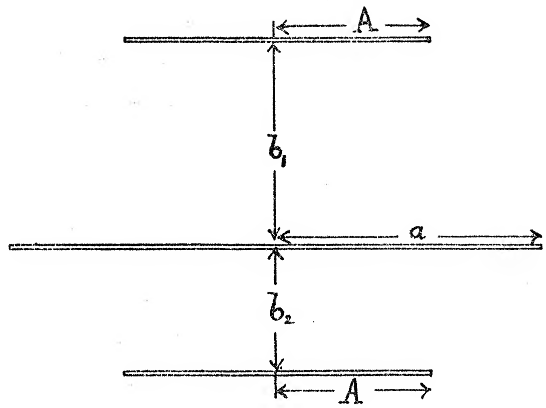

Fig. 3.

If b be chosen of such a value that the corresponding curve in fig. 2 is reasonably flat at its highest point, then the mutual inductance per turn will be practically constant over the whole section of a secondary coil whose axial and radial depths are both small; and the secondary may consist of a many-layered coil whose dimensions and position need not be known with any high accuracy. By doing this we throw all the burden of accuracy on the two primary coils which we have assumed to be mere circles. Clearly these must be replaced by single-layer coils of accurately known dimensions and relative position, and so we must extend the above investigation to the more complicated case where the primary consists of two coaxial helixes, of equal and finite length, with the secondary coil midway between them as in fig. 4. Here at the points P and R (and all round the mean circumference of ROP) the field due to the two primaries should be zero, the component lines of force due to the upper and lower helixes being tangential to one another, and in opposite directions as shown.

§ 4. *Helix and Circle.*—Several cases of the system shown in fig. 4 were investigated by means of the curves of fig. 2. The mutual inductance between the helix LN and the circle RP was taken as approximately*

$$\frac{1}{6}(M_1 + 4M_2 + M_3),$$

where M_1, M_3, and M_2 refer to circles at the ends and middle of the helix; a similar approximation to $\partial M/\partial a$ was also made and a series of curves were

* Merrifield's 'B.A. Report,' 1880.

drawn. The M and the $\partial M/\partial a$ curves for the particular dimensions which proved satisfactory are shown in fig. 5. The relative dimensions (fig. 4)

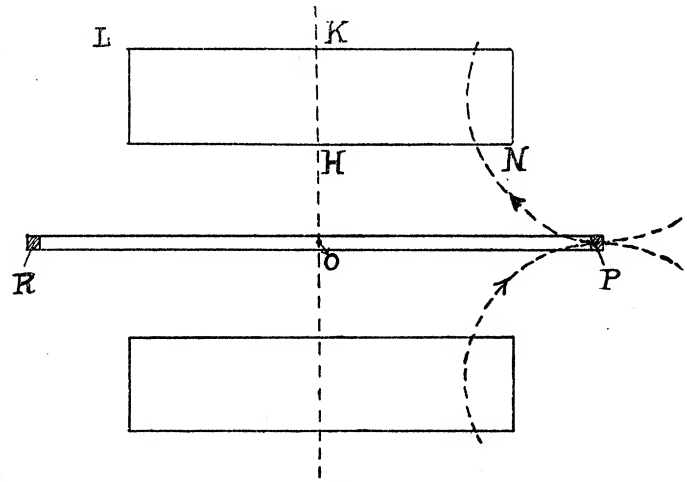

Fig. 4.

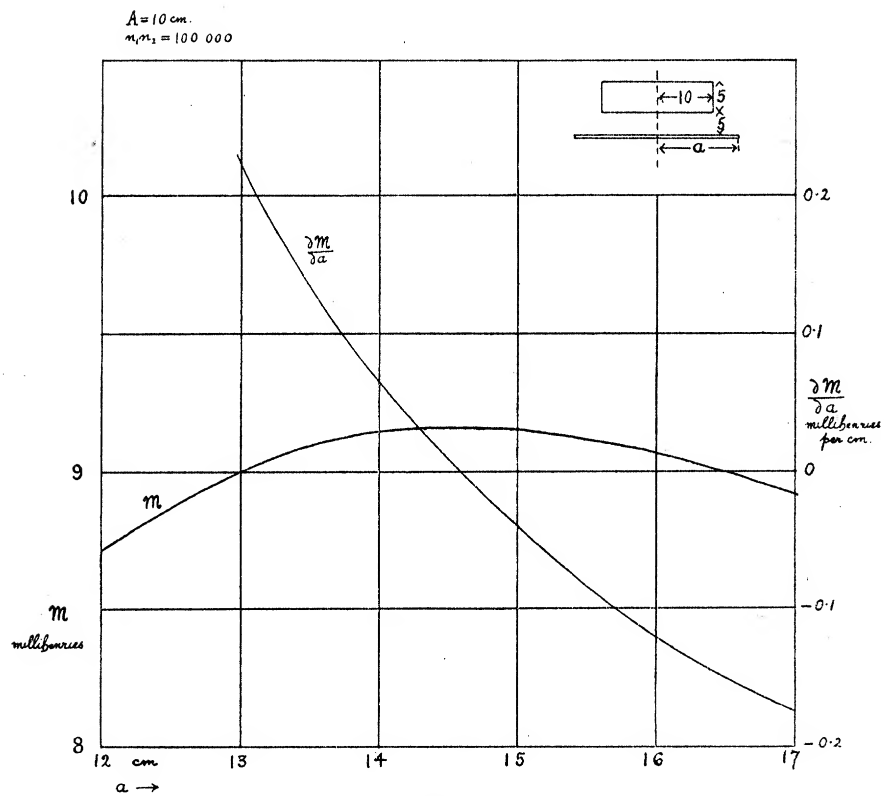

Fig. 5.

are:—HN = A = 10, OH = 5, OK = 10, and OP = 14·60, this last being the value of a for which $\partial M/\partial a = 0$.

It will be noticed that, near the value $a = 14 \cdot 60$, M shows very little variation with a. (I should mention that the values for M shown in fig. 5 for a single helix and a circle are the same for the two helixes and circle, provided the total number of primary turns be kept constant.)

After the proper relative dimensions had been found by the above method, it was thought desirable to check the results by means of the much more laborious accurate formula of Viriamu Jones.* The formula is applied to the helix LN in fig. 4 by calculating for a helix of height OK, and then for one of height OH, and taking the difference.

If
$$c = \frac{2\sqrt{Aa}}{a+A}, \qquad c' = \frac{a-A}{a+A},$$

$$k = \frac{2\sqrt{Aa}}{\sqrt{(a+A)^2+b^2}} = \sin\gamma, \qquad k' = \cos\gamma,$$

and
$$\sin\beta = c'/k',$$

then the Viriamu Jones formula may be reduced to

$$M = 2\times 10^{-9}\varpi n_1 n_2 (A+a)\left\{\frac{cF_1(k)-E_1(k)}{k} - \frac{a-A}{b}\Psi\right\}, \tag{3}$$

where

$$\Psi = -\frac{\varpi}{2} - [F_1(k)-E_1(k)]\,F(k'\beta) + F_1(k)\,E(k'\beta),$$

also
$$\frac{\partial M}{\partial a} = \frac{10^{-9}\varpi n_1 n_2 c}{A}\left\{2AkF_1(k) + \frac{(a+A)^2 c}{b}\Psi\right\}. \tag{4}$$

From (3) were calculated the values of M for A = 10, helix $b = 5$ to 10, with a = 14·1, 14·3, 14·5, and 14·7 cm. respectively. The results which are given in the following table entirely corroborate those obtained by the less exact method.

Similarly, formula (4) gave $\partial M/\partial a = 0$ for $a = 14 \cdot 60$.

Table.—(A = 10, b = 5 to 10, $n_1 n_2$ = 100,000.)

a (cm.)	14·1	14·3	14·5	14·7
Millihenries	9·1614	9·1754	9·1762	9·1763

Finally the variation in M due to a small axial displacement of the secondary coil (from the mid position) was estimated. It was found that a displacement of 0·35 cm. reduced M by less than 1 in 10,000.

* 'Roy. Soc. Proc.,' p. 192, December 9, 1897.

It will be seen that, with the above proportions, if the radius of the secondary coil is 14·6 cm., we may make it a coil of many layers and of appreciable cross-section.

If, for example, the cross-section be 0·5 cm. × 0·5 cm., then the maximum variation from the mean value, over the whole section, of the inductance per turn will be the same within a few parts in 1000, and thus we can with perfect safety obtain an accurate result by using a method of averaging, such as the Purkiss formula used by Lord Rayleigh.*

§ 5. *Actual Construction of Standard.*—A standard of the design described above is at present being constructed at the National Physical Laboratory. In this the two primary helixes are of bare wire (75 turns each) wound on one marble cylinder of 30 cm. diameter, while the secondary coil consists of 488 turns in a channel of 1 sq. cm. section with a mean diameter of 43·8 cm., the nominal value, therefore, being close to 10 millihenries.

I may remark that the principle here employed will, without doubt, be of value in other problems where accurately known mutual inductances are required.

In conclusion, I would express my thanks to Dr. R. T. Glazebrook for valued and helpful criticism of the design, and to Mr. F. E. Smith for kind advice with regard to the material construction of it.

* Maxwell's 'Elect. and Mag.,' vol. 2, p. 350, 3rd edition.